

CONNECTING ORBITS FOR NONLINEAR DIFFERENTIAL EQUATIONS AT RESONANCE

PIOTR KOKOCKI

ABSTRACT. We study the existence of orbits connecting stationary points for the first order differential equations being at resonance at infinity, where the right hand side is the perturbations of a sectorial operator. Our aim is to prove an index formula expressing the Conley index of associated semiflow with respect to appropriately large ball, in terms of special geometrical assumptions imposed on the nonlinearity. We also prove that the geometrical assumptions are generalization of well known in literature Landesman-Lazer and strong resonance conditions. Obtained index formula will be used to derive the criteria determining the existence of orbits connecting stationary points for the heat equation being at resonance at infinity.

1. INTRODUCTION

We consider nonlinear differential equations of the form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \quad t > 0 \quad (1.1)$$

where λ is a real number, $A : X \supset D(A) \rightarrow X$ is a sectorial operator on a Banach space X and $F : X^\alpha \rightarrow X$ is a continuous map. Here X^α for $\alpha \in (0, 1)$, is a fractional power space associated with A . We intend to study the existence of orbits connecting stationary points for the equation (1.1) being at *resonance at infinity*, that is, $\text{Ker}(\lambda I - A) \neq \{0\}$ and F is a bounded map. To explain this more precisely, assume that, for every initial data $x \in X^\alpha$, the equation (1.1) admits a (mild) solution $u(\cdot; x) : [0, +\infty) \rightarrow X^\alpha$ starting at x . We can define *the semiflow* $\Phi : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ given by

$$\Phi(t, x) := u(t; x) \quad \text{for } t \in [0, +\infty), x \in X^\alpha.$$

Then *the stationary point* of (1.1) is an element $x_0 \in X^\alpha$ such that $\Phi(t, x_0) = x_0$ for $t \geq 0$ and *the full solution* (or *orbit*) of this equation is a map $u : \mathbb{R} \rightarrow X^\alpha$ that satisfies the following equality

$$\Phi(s, u(t)) = u(t + s) \quad \text{for } s \geq 0, t \in \mathbb{R}.$$

We say that the full solution u connects stationary points $u_+, u_- \in X^\alpha$ provided there are sequences (t_n^+) and (t_n^-) with $t_n^\pm \rightarrow \pm\infty$ as $n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} u(t_n^+) = u_+ \quad \text{and} \quad \lim_{n \rightarrow +\infty} u(t_n^-) = u_-.$$

A tool that we will use to study this problem is a version of the Conley index for semiflows defined on metric spaces. This index was introduced initially for semiflows acting on finite dimensional vector spaces, see [4], [16]. In [13] and [15], Rybakowski extended this index theory on arbitrary metric space, which gave a rise to study the dynamics of partial differential equations.

This paper is motivated by [12], [14] and [11] where the Conley index methods were used to prove that existence of orbits connecting stationary points for the equation (1.1). However in these articles authors studied this problem assuming

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lack of resonance at infinity. In this paper, we continue to study this problem in the case of resonance at infinity. The main difficulty lies in the fact that, in the presence of resonance at infinity, the problem of existence of orbits connecting stationary points may not have solution for general nonlinearity F . This fact has been explained in detail in Remark 3.3. Therefore, our aim is to overcome this problem by new theorems determining the existence of connecting orbits for (1.1), in terms of appropriate geometrical assumptions imposed on F .

In Section 2 we briefly recall a necessary properties of Conley index that we will use in the paper, such as existence, multiplicity and homotopy invariance.

In Section 3 we formulate geometrical assumptions (G1) and (G2) (see page 5) and use them to prove our main result, *the index formula for bounded orbits*, which express the Conley index of the invariant set of Φ in sufficiently large ball, in terms of assumptions (G1) and (G2).

Finally, in Section 4 we provide applications for particular partial differential equations. First of all, in Theorems 4.3 and 4.4, we prove that if F is a Niemytzki operator associated with a map $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, then the well known in literature Landesman-Lazer and strong resonance conditions are actually particular case of assumptions (G1) and (G2). The Landesman-Lazer conditions were introduced in [10] to study the stationary points of heat equations being at resonance at infinity. There is many papers using these conditions to study the heat or telegrapher equations in the case of resonance. For instance in [3] and [1] theorems for the existence periodic solutions or stationary points were derived. Similarly, the strong resonance conditions were studied for example in [2], [18] in order to obtain the existence of stationary points and periodic orbits of heat equation. Encouraged by these results we use the abstract results obtained in Section 3 and prove effective criteria determining the existence of orbits connecting stationary points for the nonlinear heat equation, in terms of Landesman-Lazer and strong resonance conditions.

2. HOMOTOPY INDEX

In this section we gather the properties of Conley Index which are necessary in this paper. For more details see [15]. The continuous map $\Phi : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ is a semiflow on X^α provided

$$\Phi(0, x) = x \quad \text{and} \quad \Phi(t + t', x) = \Phi(t, \Phi(t', x)) \quad \text{for } t, t' \geq 0, x \in X^\alpha.$$

A map $\sigma : [-\delta_1, \delta_2] \rightarrow X^\alpha$, where $\delta_2 > 0$, $\delta_1 \geq 0$, is said to be *a solution* of Φ , if

$$\Phi(t, \sigma(s)) = \sigma(t + s) \quad \text{for } t \geq 0 \text{ and } s \in [-\delta_1, \delta_2], t + s \in [-\delta_1, \delta_2].$$

If σ is defined on \mathbb{R} , then σ is called *a full solution*. We say that the full solution σ *connects* stationary points $u_+, u_- \in X^\alpha$, if there are sequences (t_n^+) and (t_n^-) with

$$t_n^\pm \rightarrow \pm\infty, \quad u(t_n^+) \rightarrow u_+ \quad \text{and} \quad u(t_n^-) \rightarrow u_- \quad \text{as } n \rightarrow +\infty. \quad (2.1)$$

Let $K \subset X^\alpha$ be a subset. We say that K is *invariant* with respect to Φ , if for every $x \in K$ there is full solution σ of Φ such that $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset K$. If $N \subset X^\alpha$ then we define *maximal invariant set* as

$$\text{Inv}(N) := \text{Inv}(N, \Phi) := \{x \in N \mid \text{there is a solution } \sigma : \mathbb{R} \rightarrow X^\alpha \text{ of } \Phi \\ \text{such that } \sigma(0) = x \text{ and } \sigma(\mathbb{R}) \subset N\}.$$

A closed invariant set K is called *isolated*, if there is a closed set $N \subset X^\alpha$ such that

$$K = \text{Inv}(N) \subset \text{int } N.$$

In this case N is called *isolating neighborhood* for K . A subset $N \subset X^\alpha$ is *admissible* with respect to Φ , if for every sequences (x_n) in X^α , (t_n) in $[0, +\infty)$ such that

$t_n \rightarrow +\infty$ when $n \rightarrow \infty$, the inclusion

$$\Phi([0, t_n] \times \{x_n\}) \subset N \quad \text{for } n \geq 1,$$

implies that the set $\{\Phi(t_n, x_n) \mid n \geq 1\}$ is relatively compact in X^α .

Let $\{\Phi^s\}_{s \in [0,1]}$ be a family of semiflows where $\Phi^s : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$, $s \in [0, 1]$. We say that $N \subset X^\alpha$ is *admissible* with respect to $\{\Phi^s\}_{s \in [0,1]}$, if for every sequences $s_n \in [0, 1]$, (x_n) in X^α and (t_n) in $[0, +\infty)$ such that $t_n \rightarrow +\infty$ when $n \rightarrow \infty$, the inclusion

$$\Phi_{s_n}([0, t_n] \times \{x_n\}) \subset N \quad \text{for } n \geq 1,$$

implies that the set $\{\Phi_{s_n}^{s_n}(t_n, x_n) \mid n \geq 1\}$ is relatively compact in X^α . Furthermore the family $\{\Phi^s\}_{s \in [0,1]}$ is continuous provided $\Phi^{s_n}(t_n, x_n) \rightarrow \Phi^{s_0}(t_0, x_0)$ as long as $s_n \rightarrow s_0$, $t_n \rightarrow t_0$ and $x_n \rightarrow x_0$ as $n \rightarrow +\infty$.

From now on we write $\mathcal{S}(X^\alpha) = \mathcal{S}(X^\alpha, \Phi)$ for a class of invariant sets admitting an admissible isolated neighborhood. A special case of isolated neighborhood is an *isolating block*. To define it assume that $B \subset X^\alpha$ is a closed set and let $x \in \partial B$. We say that x is a *strict egress point* (resp. *strict ingress point*, resp. *bounce off point*), if for any solution $\sigma : [-\delta_1, \delta_2) \rightarrow X^\alpha$, where $\delta_1 \geq 0$ and $\delta_2 > 0$, of the semiflow Φ^s such that $\sigma(0) = x$ the following holds:

- (a) there is $\varepsilon_2 \in (0, \delta_2]$ such that $\sigma(t) \notin B$ (resp. $\sigma(t) \in \text{int } B$, resp. $\sigma(t) \notin B$) for $t \in (0, \varepsilon_2]$;
- (b) if $\delta_1 > 0$ then there is $\varepsilon_1 \in (0, \delta_1)$ such that $\sigma(t) \in \text{int } B$ (resp. $\sigma(t) \notin B$, resp. $\sigma(t) \notin B$) for $t \in [-\varepsilon_1, 0)$.

We write B^e , B^i and B^b for the sets of strict egress points, strict ingress points and bounce off points, respectively. Furthermore, put $B^- := B^e \cup B^b$. A close set $B \subset X^\alpha$ is an *isolating block*, if $\partial B = B^e \cup B^i \cup B^e$ and the set B^- is closed.

For $K \in \mathcal{S}(X^\alpha)$ we define the Conley index (*the homotopy index*) of K as the homotopy type $h(\Phi^s, K)$ of pointed space given by

$$h(\Phi^s, K) := \begin{cases} [B/B^-, [B^-]] & \text{if } B^- \neq \emptyset; \\ [B \dot{\cup} \{c\}, c] & \text{if } B^- = \emptyset \end{cases}$$

where B/B^- is the quotient space and $B \dot{\cup} \{c\}$ is a disjoint sum of B and the one point space $\{c\}$. It is known that the homotopy index is independent from the choice of isolating block of K has the following properties:

- (H1) (*Existence*) If $K \in \mathcal{S}(X^\alpha)$ and $h(\Phi, K) \neq \bar{0}$, then $K \neq \emptyset$.
- (H2) (*Addition*) If $K_1, K_2 \in \mathcal{S}(X^\alpha)$ are disjoint sets, then $K_1 \cup K_2 \in \mathcal{S}(X^\alpha)$ and
$$h(\Phi, K_1 \cup K_2) = h(\Phi, K_1) \vee h(\Phi, K_2).$$
- (H3) (*Multiplication*) Let $\Phi^i : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$, for $i = 1, 2$ be semiflows. If $K_1, K_2 \in \mathcal{S}(X^\alpha)$ then $K_1 \times K_2 \in \mathcal{S}(\Phi^1 \times \Phi^2, X^\alpha \times X^\alpha)$ and
$$h(\Phi^1 \times \Phi^2, K_1 \times K_2) = h(\Phi^1, K_1) \wedge h(\Phi^2, K_2).$$
- (H4) (*Homotopy invariance*) Let $\{\Phi^s : [0, +\infty) \times X^\alpha \rightarrow X^\alpha\}_{s \in [0,1]}$ be a continuous family of semiflows and let the set $N \subset X$ be admissible with respect to this family. If for any $s \in [0, 1]$ the set N is an isolating neighborhood of $K_s := \text{Inv}(\Phi^s, N)$, then $K_s \in \mathcal{S}(\Phi^s, X^\alpha)$ for $s \in [0, 1]$ and

$$h(\Phi^0, \text{Inv}(\Phi^0, N)) = h(\Phi^1, \text{Inv}(\Phi^1, N)).$$

3. INDEX FORMULA FOR BOUNDED ORBITS

We will study the problem of existence of orbits connecting stationary points for the equations of the form

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(u(t)), \quad t > 0, \quad (3.1)$$

where λ is an eigenvalue of a sectorial operator $A : X \supset D(A) \rightarrow X$ on a real Banach space X with norm $\|\cdot\|$ and $F : X^\alpha \rightarrow X$ is a continuous map. Here X^α for $\alpha \in (0, 1)$, is a fractional space given by $X^\alpha := D((A + \delta I)^\alpha)$, where $\delta > 0$ is such that $A + \delta I$ is a positively defined operator. We assume that

- (A1) A has compact resolvent,
- (A2) there exists a Hilbert space H equipped with a scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$ with a continuous injection $i : X \hookrightarrow H$,
- (A3) there exists a linear self-adjoint operator $\hat{A} : H \supset D(\hat{A}) \rightarrow H$ with the property that $\text{Gr}(A) \subset \text{Gr}(\hat{A})$, where the inclusion is understood by the map $X \times X \xrightarrow{i \times i} H \times H$.
- (F1) given $x \in X^\alpha$ there is an open $V \subset X^\alpha$ of x and $L > 0$ such that
$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|_\alpha \quad \text{for } x_1, x_2 \in V;$$
- (F2) there is a constant $m > 0$ such that $\|F(x)\| \leq m$ for $x \in X^\alpha$.

Let $\{S_A(t)\}_{t \geq 0}$ be a semigroup generated by $-A$ and let $J \subset \mathbb{R}$ be an interval. We say that a continuous map $u : J \rightarrow X^\alpha$ is a *mild solution* of equation (3.1), if

$$u(t) = S_A(t - t')u(t') + \int_{t'}^t S_A(t - \tau)F(s, u(\tau)) d\tau$$

for every $t, t' \in J$, $t' < t$. Since A is sectorial and assumptions (F1) and (F2) hold, it is known (see [8, Theorem 3.3.3, Corollary 3.3.5]) that for any $x \in X^\alpha$ equation (3.1) admits a unique mild solution $u(t; x) : [0, +\infty) \rightarrow X^\alpha$ starting at x . Hence we define a semiflow $\Phi : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ associated with this equation by

$$\Phi(t, x) := u(t; x) \quad \text{for } t \geq 0, x \in X^\alpha.$$

In view of Theorems 3.4 and 3.5 from [9], the semiflow Φ is continuous and admissible with respect to any bounded set $N \subset X^\alpha$.

Remark 3.1. The spectrum $\sigma(A)$ consists of the sequence (possibly finite) of real eigenvalues. Indeed, the operator A has compact resolvents which implies that

$$\sigma(A) = \{\lambda_i \mid i \geq 1\} \subset \mathbb{C}$$

and this set is finite or $|\lambda_i| \rightarrow +\infty$ when $n \rightarrow +\infty$. Furthermore, if $\lambda \in \mathbb{C}$ is a complex eigenvalue of A , then, by (A3), it is also a complex eigenvalue of the symmetric operator \hat{A} and hence λ is a real number. \square

From the above remark it follows that the spectrum $\sigma(A)$ of the operator A consists of the sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_i < \lambda_{i+1} < \dots$$

which is finite or $\lambda_i \rightarrow +\infty$ when $i \rightarrow +\infty$. Furthermore, by Theorem 2.1 from [9] we have the following

Theorem 3.2. *Under the assumptions (A1), (A2) and (A3), if $\lambda = \lambda_k$ for some $k \geq 1$, is an eigenvalue of A , then there exists a decomposition $X = X_+ \oplus X_- \oplus X_0$ on closed subspaces, such that*

$$S_A(t)X_i \subset X_i \quad \text{for } t \geq 0, i \in \{0, -, +\}, \quad (3.2)$$

and the following holds:

- (i) $X_0 = \text{Ker}(\lambda I - A)$ and the space X_- is such that

$$X_- = \{0\} \text{ provided } k = 1 \text{ and } X_- = \bigoplus_{i=1}^{k-1} \text{Ker}(\lambda_i I - A) \text{ provided } k \geq 2.$$

Therefore $\dim X_- = 0$ for $k = 1$ and $\dim X_- = \sum_{i=1}^{k-1} \dim \text{Ker}(\lambda_i I - A)$ for $k \geq 2$.

(ii) there are $c, M > 0$ with the property that

$$\|A_\delta^\alpha S_A(t)x\| \leq M e^{-(\lambda+c)t} t^{-\alpha} \|x\| \quad \text{if } t > 0, x \in X_+, \quad (3.3)$$

$$\|e^{\lambda t} S_A(t)x\| \leq M e^{-ct} \|x\| \quad \text{if } t \geq 0, x \in X_+, \quad (3.4)$$

$$\|e^{\lambda t} S_A(t)x\| \leq M e^{ct} \|x\| \quad \text{if } t \leq 0, x \in X_-, \quad (3.5)$$

(iii) X_0, X_- and X_+ are mutually orthogonal spaces, that is,

$$\langle i(u_l), i(u_m) \rangle_H = 0 \quad \text{for } u_l \in X_l \text{ and } u_m \in X_m \quad (3.6)$$

where $l, m \in \{0, -, +\}, l \neq m$.

Assume that $P, Q_\pm : X \rightarrow X$ are projections given

$$Px = x_0 \text{ and } Q_\pm x = x_\pm \quad \text{for } x \in X \quad (3.7)$$

where $x = x_+ + x_0 + x_-$ if $x_i \in X_i, i \in \{0, -, +\}$. Put $Q := Q_- + Q_+$. In view of the continuity of the inclusion $X^\alpha \subset X$, we have $X^\alpha = X_0 \oplus X_-^\alpha \oplus X_+^\alpha$, where

$$X_-^\alpha := X^\alpha \cap X_-, \quad X_+^\alpha := X^\alpha \cap X_+$$

are closed spaces. Hence P and Q_\pm can be seen as continuous maps $P, Q_\pm : X^\alpha \rightarrow X^\alpha$. Observe also that (3.2) implies

$$S_A(t)Px = PS_A(t)x \text{ and } S_A(t)Q_\pm x = Q_\pm S_A(t)x \quad \text{for } t \geq 0, x \in X. \quad (3.8)$$

Remark 3.3. If the equation (3.1) is at resonance at infinity the problem of existence of orbits connecting stationary points may not have solution for general nonlinearity F . To see this it is enough to take $F(x) = y_0$ for $x \in X$, where $y_0 \in \text{Ker}(\lambda I - A) \setminus \{0\}$. Indeed, if $u : \mathbb{R} \rightarrow X^\alpha$ is an orbit connecting stationary points for equation (3.1), then

$$u(t) = e^{\lambda(t-t')} S_A(t-t')u(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t-\tau)y_0 d\tau \quad \text{for } t > t'.$$

Since $\text{Ker}(\lambda I - A) \subset \text{Ker}(I - e^{\lambda t} S_A(t))$ for $t \geq 0$ it follows that

$$u(t) = e^{\lambda(t-t')} S_A(t-t')u(t') + (t-t')y_0 \quad \text{for } t > t'.$$

Therefore by (3.8)

$$Pu(t) = e^{\lambda(t-t')} S_A(t-t')Pu(t') + (t-t')Py_0 = Pu(t') + (t-t')y_0 \quad \text{for } t > t',$$

and hence $Pu(h) = Pu(0) + hy_0$ for $h \geq 0$. Let sequences $(t_n^+), (t_n^-)$ and stationary points $u_+, u_- \in X^\alpha$ be such that (2.1) holds. Putting $h := t_n^+$ we obtain a contradiction because $y_0 \neq 0$.

To overcome the difficulty discussed in the above remark we shall introduce geometric conditions for F which will guarantee the existence of bounded orbits for equation (3.1):

$$(G1) \quad \begin{cases} \text{for every ball } B \subset X_+^\alpha \oplus X_-^\alpha \text{ there is } R > 0 \text{ such that} \\ \langle F(x+y), x \rangle_H > 0 \text{ for } (x, y) \in X_0 \times B \text{ such that } \|x\|_H \geq R, \end{cases}$$

$$(G2) \quad \begin{cases} \text{for every ball } B \subset X_+^\alpha \oplus X_-^\alpha \text{ there is } R > 0 \text{ such that} \\ \langle F(x+y), x \rangle_H < 0 \text{ for } (x, y) \in X_0 \times B \text{ such that } \|x\|_H \geq R. \end{cases}$$

Now we proceed to the main result of this section, namely *the index formula for bounded orbits*. It is a tool to determining the Conley index for the maximal invariant set contained in appropriately large ball in terms of geometrical conditions (G1) and (G2). This theorem can be used directly to prove the existence of bounded

orbits for the equation (3.1) or can be applied to prove the existence of orbits connecting stationary points, which is studying in the subsequent theorem. Write

$$d_0 := 0 \quad \text{and} \quad d_l := \sum_{i=1}^l \dim \text{Ker}(\lambda_i I - A) \quad \text{for} \quad l \geq 1.$$

Theorem 3.4. *If $\lambda = \lambda_k$ for some $k \geq 1$, then there is a closed isolated neighborhood $N \subset X^\alpha$, admissible with respect to the semiflow Φ such that $0 \in \text{int } N$ and, for $K := \text{Inv}(N, \Phi)$, the following statements hold:*

- (a) *if condition (G1) is satisfied, then $h(\Phi, K) = \Sigma^{d_k}$,*
- (b) *if condition (G2) is satisfied, then $h(\Phi, K) = \Sigma^{d_{k-1}}$.*

In the proof of the above theorem we will consider the family of equations

$$\dot{u}(t) = -Au(t) + \lambda u(t) + G(s, u(t)), \quad t > 0. \quad (3.9)$$

where $G : [0, 1] \times X^\alpha \rightarrow X$ is a map given by

$$G(s, x) := PF(sQx + Px) + sQF(sQx + Px) \quad \text{for} \quad (s, x) \in [0, 1] \times X^\alpha. \quad (3.10)$$

Since F is locally lipschitz, it is not difficult to check that $G(s, \cdot)$ satisfies condition (F1) for every $s \in [0, 1]$. Furthermore by (F2), for any $s \in [0, 1]$ and $x \in X^\alpha$ we have

$$\begin{aligned} \|G(s, x)\| &= \|PF(sQx + Px) + sQF(sQx + Px)\| \\ &\leq \|P\| \|F(sQx + Px)\| + \|Q\| \|F(sQx + Px)\| \\ &\leq m(\|P\| + \|Q\|) := m_0, \end{aligned} \quad (3.11)$$

which shows that condition (F2) is satisfied. Therefore, for any $s \in [0, 1]$, one can define a semiflow $\Psi^s : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ given by the formula

$$\Psi^s(t, x) := u(t; s, x) \quad \text{for} \quad t \in [0, +\infty), \quad x \in X^\alpha,$$

where $u(\cdot; s, x) : [0, +\infty) \rightarrow X^\alpha$ is a weak solution of (3.9) starting at x . Theorems 3.4 and 3.5 from [9] say that the family $\{\Psi^s\}_{s \in [0, 1]}$ is continuous and any bonded set is admissible. Hence we have a homotopy between $\Psi^1 = \Phi$ and the semiflow Ψ^0 associated with

$$\dot{u}(t) = -Au(t) + \lambda u(t) + PF(Pu(t)), \quad t > 0.$$

Note that every solution $u : [0, +\infty) \rightarrow X^\alpha$ of this equation satisfies the formula

$$u(t) = e^{\lambda t} S_A(t)u(0) + \int_0^t PF(Pu(\tau)) d\tau \quad \text{for} \quad t \geq 0 \quad (3.12)$$

Let $\psi_1 : [0, +\infty) \times X_-^\alpha \oplus X_+^\alpha \rightarrow X_-^\alpha \oplus X_+^\alpha$ be a semiflow given by

$$\psi_1(t, x) := e^{\lambda t} S_A(t)x \quad \text{for} \quad t \in [0, +\infty), \quad x \in X_-^\alpha \oplus X_+^\alpha$$

and let semiflow $\psi_2 : [0, +\infty) \times X_0 \rightarrow X_0$ be associated with the equation

$$\dot{u}(t) = PF(u(t)), \quad t > 0.$$

Then it is easy to see that

$$\Psi^0(t, x) = \psi_1(t, Qx) + \psi_2(t, Px) \quad \text{for} \quad t \in [0, +\infty), \quad x \in X^\alpha,$$

and therefore the semiflow Ψ^0 is equivalent with the product of ψ_1 and ψ_2 , that is, for any $t \geq 0$, and $(x, y) \in (X_-^\alpha \oplus X_+^\alpha) \times X_0$ we have

$$\Psi^0(t, U(x, y)) = U(\psi_1(t, x), \psi_2(t, y)) \quad (3.13)$$

where $U : (X_-^\alpha \oplus X_+^\alpha) \times X_0 \rightarrow X^\alpha$ is defined by $U(x, y) = x + y$ for $(x, y) \in (X_-^\alpha \oplus X_+^\alpha) \times X_0$.

In the first step we prove the following lemma, which provides some *a priori* bounds for solutions of the equation (3.9).

Lemma 3.5. *There is $R > 0$ such that for any $s \in [0, 1]$ and for any bounded full solution $u = u_s : \mathbb{R} \rightarrow X^\alpha$ for the semiflow Ψ^s , the following inequality holds*

$$\|Qu(t)\|_\alpha \leq R \quad \text{for } t \in \mathbb{R}.$$

Proof. Let $s \in [0, 1]$ be fixed and let $u = u_s : \mathbb{R} \rightarrow X^\alpha$ be a full solution for the equation (3.9). Since $Q_-, Q_+ : X^\alpha \rightarrow X^\alpha$ are bounded operators, the sets $\{Q_+u(t) \mid t \leq 0\}$ and $\{Q_-u(t) \mid t \geq 0\}$ are bounded in X^α . We prove that

$$\|Q_+u(t)\|_\alpha \leq m_0 M \|Q_+\|_{L(X)} \left(e^{-c}/c + \frac{1}{1-\alpha} \right) =: R_1 \quad \text{for } t \in \mathbb{R}. \quad (3.14)$$

Indeed, since u is full solution, we have the equality $\Psi^s(t - t', u(t')) = u(t)$ for $t, t' \in \mathbb{R}$, $t \geq t'$, which implies that

$$u(t) = e^{\lambda(t-t')} S_A(t - t') u(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t - \tau) G(s, u(\tau)) d\tau \quad (3.15)$$

for $t \geq t'$. By (3.8) one find that

$$Q_+u(t) = e^{\lambda(t-t')} S_A(t - t') Q_+u(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t - \tau) Q_+G(s, u(\tau)) d\tau \quad (3.16)$$

for $t \geq t'$. Since X^α embeds continuously in X , there is a constant $C > 0$ such that $\|x\| \leq C\|x\|_\alpha$ for $x \in X^\alpha$. Furthermore, by the inequality (3.3), there are constants $c, M > 0$ such that

$$\begin{aligned} \|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha &= \|A_\delta^\alpha e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\| \\ &\leq M \frac{e^{-c(t-t')}}{(t-t')^\alpha} \|Q_+u(t')\| \leq CM \frac{e^{-c(t-t')}}{(t-t')^\alpha} \|Q_+u(t')\|_\alpha \end{aligned}$$

for $t, t' \in \mathbb{R}$, $t > t'$. Hence, the boundedness of $\{Q_+u(t) \mid t \leq 0\}$ implies that

$$\|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha \rightarrow 0 \quad \text{as } t' \rightarrow -\infty. \quad (3.17)$$

In view of the inequality (3.3), we deduce that

$$\begin{aligned} \|Q_+u(t)\|_\alpha &\leq \|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha \\ &\quad + \int_{t'}^t \|A_\delta^\alpha e^{\lambda(t-\tau)} S_A(t - \tau) Q_+G(s, u(\tau))\| d\tau \\ &\leq \|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha + M \int_{t'}^t \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} \|Q_+G(s, u(\tau))\| d\tau \\ &\leq \|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha + m_0 M \|Q_+\|_{L(X)} \int_{t'}^t \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} d\tau. \end{aligned} \quad (3.18)$$

Furthermore, taking $t, t' \in \mathbb{R}$ such that $t - t' > 1$, one find that

$$\begin{aligned} \int_{t'}^t \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} d\tau &= \int_{t'}^{t-1} \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} d\tau + \int_{t-1}^t \frac{e^{-c(t-\tau)}}{(t-\tau)^\alpha} d\tau \\ &\leq \int_{t'}^{t-1} e^{-c(t-\tau)} d\tau + \int_{t-1}^t \frac{1}{(t-\tau)^\alpha} d\tau = (e^{-c} - e^{c(t'-t)})/c + 1/(1-\alpha), \end{aligned}$$

and hence, by (3.18), it follows that

$$\begin{aligned} \|Q_+u(t)\|_\alpha &\leq \|e^{\lambda(t-t')} S_A(t - t') Q_+u(t')\|_\alpha \\ &\quad + m_0 M \|Q_+\|_{L(X)} \left((e^{-c} - e^{c(t'-t)})/c + \frac{1}{1-\alpha} \right). \end{aligned}$$

Using (3.17) and passing to limit with $t' \rightarrow -\infty$ we infer that (3.14) is satisfied. Since X_- is finite dimensional there is constant $C' > 0$ such that

$$\|x\|_\alpha \leq C'\|x\| \quad \text{for } x \in X_-. \quad (3.19)$$

We show that the boundedness of $\{Q_-u(t) \mid t \geq 0\}$ in X^α implies

$$\|Q_-u(t)\|_\alpha \leq m_0 C' M \|Q_-\|_{L(X)} / c := R_2 \quad \text{for } t \in \mathbb{R}. \quad (3.20)$$

In order to get this inequality we apply the operator Q_- on equation (3.15) and, by (3.8), we obtain

$$Q_-u(t) = e^{\lambda(t-t')} S_A(t-t') Q_-u(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t-\tau) Q_-G(s, u(\tau)) d\tau.$$

In consequence, for $t, t' \in \mathbb{R}$ and $t \geq t'$, we have

$$e^{\lambda(t'-t)} S_A(t'-t) Q_-u(t) = Q_-u(t') + \int_{t'}^t e^{\lambda(t'-\tau)} S_A(t'-\tau) Q_-G(s, u(\tau)) d\tau, \quad (3.21)$$

where we used the fact that the family $\{S_A(t)\}_{t \geq 0}$ extends on the space X_- to the C_0 group of bounded operators. The inequality (3.5) implies that

$$\begin{aligned} \|Q_-u(t')\| &\leq \|e^{\lambda(t'-t)} S_A(t'-t) Q_-u(t)\| \\ &\quad + \int_{t'}^t \|e^{\lambda(t'-\tau)} S_A(t'-\tau) Q_-G(s, u(\tau))\| d\tau \\ &\leq \|e^{\lambda(t'-t)} S_A(t'-t) Q_-u(t)\| + \int_{t'}^t M e^{c(t'-\tau)} \|Q_-G(s, u(\tau))\| d\tau \\ &\leq M e^{c(t'-t)} \|Q_-u(t)\| + m_0 M \|Q_-\|_{L(X)} \int_{t'}^t e^{c(t'-\tau)} d\tau \\ &\leq C M e^{c(t'-t)} \|Q_-u(t)\|_\alpha + m_0 M \|Q_-\|_{L(X)} (1 - e^{c(t'-t)}) / c. \end{aligned}$$

Therefore after passing to the limit with $t \rightarrow +\infty$ we infer that

$$\|Q_-u(t')\| \leq m_0 M \|Q_-\| / c \quad \text{for } t' \in \mathbb{R} \quad (3.22)$$

which along with (3.19) gives (3.20). Hence, combining (3.14) and (3.20) gives

$$\begin{aligned} \|Qu(t)\|_\alpha &\leq \|Q_-u(t)\|_\alpha + \|Q_+u(t)\|_\alpha \leq \|Q_-\|_{L(X^\alpha)} \|u(t)\|_\alpha + \|Q_+\|_{L(X^\alpha)} \|u(t)\|_\alpha \\ &\leq R_1 \|Q_+\|_{L(X^\alpha)} + R_2 \|Q_-\|_{L(X^\alpha)} := R, \end{aligned}$$

for $t \in \mathbb{R}$, which completes the proof. \square

Proof of Theorem 3.4. Step 1. Proposition 3.5 says that there is a constant $R_1 > 0$ such that for any $s \in [0, 1]$ and for any full solution $u = u_s : \mathbb{R} \rightarrow X^\alpha$ for the semiflow Ψ^s which is bounded in X^α we have

$$\|Qu(t)\|_\alpha \leq R_1 \quad \text{for } t \in \mathbb{R}. \quad (3.23)$$

Suppose that condition (G1) is satisfied. If we take $N_1 := \{x \in X_+^\alpha \oplus X_+^\alpha \mid \|x\|_\alpha \leq R_1 + 1\}$, then by (3.6), there is $R_2 > 0$ such that

$$\langle PF(x+y), x \rangle_H > 0 \quad \text{for } (y, x) \in N_1 \times X_0 \text{ with } \|x\|_H \geq R_2. \quad (3.24)$$

Similarly, if we assume that condition (G2) is satisfied, then there is $R_2 > 0$ such that

$$\langle PF(x+y), x \rangle_H < 0 \quad \text{for } (y, x) \in N_1 \times X_0 \text{ with } \|x\|_H \geq R_2. \quad (3.25)$$

Define $N_2 := \{x \in X_0 \mid \|x\|_H \leq R_2\}$.

Step 2. We claim that $N := N_1 \oplus N_2$ is an isolating neighborhood for the family $\{\Psi^s\}_{s \in [0, 1]}$. To proof this, let $u := u_s : \mathbb{R} \rightarrow X^\alpha$ be a full solution for the semiflow

Ψ^s , where $s \in [0, 1]$, such that $u(\mathbb{R}) \subset N$ and $u(\mathbb{R}) \cap \partial N \neq \emptyset$. Without loss of generality one can assume that $u(0) \in \partial N$. Then we have either $\|Qu(0)\|_\alpha = R_1 + 1$ and $\|Pu(0)\|_H \leq R_2$ or $\|Qu(0)\|_\alpha \leq R_1 + 1$ and $\|Pu(0)\|_H = R_2$. From (3.23), it follows that the later holds. Since u satisfies the integral formula

$$u(t) = e^{\lambda(t-t')} S_A(t-t')u(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t-\tau)G(s, u(\tau)) d\tau \quad \text{for } t \geq t',$$

from (3.8) we obtain

$$Pu(t) = e^{\lambda(t-t')} S_A(t-t')Pu(t') + \int_{t'}^t e^{\lambda(t-\tau)} S_A(t-\tau)PG(s, u(\tau)) d\tau. \quad (3.26)$$

On the other hand $\text{Ker}(\lambda I - A) \subset \text{Ker}(I - e^{\lambda t} S_A(t))$ for $t \geq 0$, and therefore (3.26) takes the form

$$Pu(t) = Pu(t') + \int_{t'}^t PF(sQu(\tau) + Pu(\tau)) d\tau.$$

Endowing X_0 with norm $\|\cdot\|_H$ we see that the map $\mathbb{R} \ni t \rightarrow Pu(t) \in X_0$ is continuously differentiable on \mathbb{R} and furthermore

$$\frac{d}{dt}\|Pu(t)\|_H^2 = 2 \left\langle \frac{d}{dt}Pu(t), Pu(t) \right\rangle_H = 2 \langle PF(sQu(t) + Pu(t)), Pu(t) \rangle_H$$

for $t \in \mathbb{R}$. Since $\|Pu(0)\|_H^2 = R_2$, combining (3.23), (3.24) and (3.25) we infer that

$$\frac{d}{dt}\|Pu(t)\|_H^2|_{t=0} = 2 \langle PF(sQu(0) + Pu(0)), Pu(0) \rangle_H \neq 0$$

if either (G1) or (G2) is satisfied. Therefore we find that the set $\{Pu(t) \mid t \in \mathbb{R}\}$ is not contained in N_2 , which contradicts the inclusion $\{u(t) \mid t \in \mathbb{R}\} \subset N$.

Step 3. Now we verify that $B := N_2$ is an isolating block for the semiflow ψ_2 and

$$\begin{cases} (B, B^-) = (N_2, \partial N_2) & \text{if condition (G1) is satisfied;} \\ (B, B^-) = (N_2, \emptyset) & \text{if condition (G2) is satisfied.} \end{cases} \quad (3.27)$$

Assume that $u : [-\delta_2, \delta_1) \rightarrow X_0$, where $\delta_1 > 0$, $\delta_2 \geq 0$, is a solution for ψ_2 such that $u(0) \in \partial N_2$. Then

$$u(t) = u(0) + \int_0^t PF(u(\tau)) d\tau \quad \text{for } t \in [-\delta_2, \delta_1),$$

which implies that the map $[-\delta_2, \delta_1) \ni t \rightarrow u(t) \in X_0$ is continuously differentiable when the space X_0 is endowed with the norm $\|\cdot\|_H$ and furthermore

$$\frac{d}{dt}\|u(t)\|_H^2 = 2 \langle \dot{u}(t), u(t) \rangle_H = 2 \langle PF(u(t)), u(t) \rangle_H \quad \text{for } t \in [-\delta_2, \delta_1).$$

Since $\|u(0)\|_H = R_2$, combining (3.24) and (3.25) gives

$$\begin{cases} \frac{d}{dt}\|u(t)\|_H^2|_{t=0} > 0 & \text{if condition (G1) is satisfied;} \\ \frac{d}{dt}\|u(t)\|_H^2|_{t=0} < 0 & \text{if condition (G2) is satisfied.} \end{cases} \quad (3.28)$$

The first inequality implies that in the case of (G1) the pair $(B, B^-) := (N_2, \partial N_2)$ is an isolating block for the semiflow ψ_2 . Similarly, in the case of the condition (G2), the second inequality (3.28) shows that the pair (N_2, \emptyset) is an isolating block for the semiflow ψ_2 .

Step 4. Applying Step 1 and homotopy invariance of Conley index we infer that

$$h(\Phi, K) = h(\Psi^1, K_1) = h(\Psi^0, K_0) \quad (3.29)$$

where $K_s := \text{Inv}(N, \Psi^s)$ for $s \in \{0, 1\}$. Further, combining (3.4), (3.5) and the fact that

$$(\delta I + A)^\alpha S_A(t)x = S_A(t)(\delta I + A)^\alpha x \quad \text{for } x \in X^\alpha,$$

we deduce that there are constants $c, M > 0$ such that

$$\begin{aligned} \|e^{\lambda t} S_A(t)x\|_\alpha &\leq M e^{-ct} \|x\|_\alpha & \text{for } t \geq 0, x \in X_+, \\ \|e^{\lambda t} S_A(t)x\|_\alpha &\leq M e^{ct} \|x\|_\alpha & \text{for } t \leq 0, x \in X_-. \end{aligned}$$

Hence [15, Theorem 11.1] shows that N_1 is an isolating neighborhood for ψ_1 and $K_0^1 := \text{Inv}(\psi_1, N_1) = \{0\}$ with

$$h(\psi_1, K_0^1) = \Sigma^{\dim X_-} = \Sigma^{d_{k-1}}. \quad (3.30)$$

In view of Step 3, the set N_2 is an isolating block for ψ_2 . Take $K_0^2 := \text{Inv}(\psi_2, N_2)$. Combining (3.13), (3.29) and multiplication property of Conley Index we infer that

$$h(\Phi, K) = h(\Psi^0, K_0) = h(\psi_1, K_0^1) \wedge h(\psi_2, K_0^2).$$

Therefore, by (3.30), we have

$$h(\Phi, K) = \Sigma^{\dim X_-} \wedge h(\psi_2, K_0^2) = \Sigma^{d_{k-1}} \wedge h(\psi_2, K_0^2). \quad (3.31)$$

In the case of condition (G1), the set N_2 is a ball in the Hilbert space $(X_0, \|\cdot\|_H)$ and hence the pair (B, B^-) is homeomorphic with the pair of topological spaces where the first is a unit ball in the Euclidean space and the second space is its boundary (to see this it is enough to take orthogonal base in X_0). In consequence

$$h(\psi_2, K_0^2) = \Sigma^{\dim X_0}.$$

Substituting this in (3.31) we deduce that

$$h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^{\dim X_0} = \Sigma^{d_k}$$

which proves point (i). In the case of condition (G2), from (3.27) it follows that the pair (N_2, \emptyset) is an isolating block for the semiflow ψ_2 , which yields $h(\psi_2, K_0^2) = \Sigma^0$. Hence, by (3.31), we infer that

$$h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^0 = \Sigma^{d_{k-1}}$$

which completes the proof of point (ii). \square

Now we apply Theorem 3.4 to study the orbits connecting stationary points. Before we do this, we make the following additional assumption on F :

(F4) $F(0) = 0$, and the map F is differentiable at 0 and there is $\mu \in \mathbb{R}$ such that

$$DF(0)[x] = \mu x \quad \text{for } x \in X^\alpha.$$

From (F4) it follows that 0 is a stationary point for the semiflow Φ , that is $\Phi(t, 0) = 0$ for $t \geq 0$. The following theorem is a criterion determining the existence of orbits connecting stationary points for equation (3.1).

Theorem 3.6. *Let $\lambda = \lambda_k$ for some $k \geq 1$ and assume that one of the following conditions is satisfied*

- (i) (G1) holds and $\lambda_l < \lambda + \nu < \lambda_{l+1}$ where $\lambda_l \neq \lambda$;
- (ii) (G1) holds and $\lambda + \nu < \lambda_1$;
- (iii) (G2) holds, $\lambda_{l-1} < \lambda + \nu < \lambda_l$ and $\lambda \neq \lambda_l$, where $l \geq 2$;
- (iv) (G2) hold, $\lambda + \nu < \lambda_1$ and $\lambda \neq \lambda_1$.

Then there is a full nonzero solution $\sigma : \mathbb{R} \rightarrow X^\alpha$ for the semiflow Φ such that

$$\lim_{t \rightarrow -\infty} u(t) = 0 \quad \text{or} \quad \lim_{t \rightarrow +\infty} u(t) = 0.$$

In the proof of this theorem we need the following lemma.

Lemma 3.7. (see [15, Theorem 3.5]) *If $\lambda + \mu \notin \sigma(A)$, then $\{0\} \in \mathcal{S}(\Phi, X^\alpha)$ and $h(\Phi, \{0\}) = \Sigma^{b_l}$, where $b_l := 0$ if $\lambda + \mu < \lambda_1$ and*

$$b_l := \sum_{i=1}^l \dim \text{Ker}(\lambda_i I - A) \quad \text{if} \quad \lambda_l < \lambda + \mu < \lambda_{l+1}.$$

Proof of the Theorem 3.6. From Lemma 3.7, it follows that $\{0\} \in \mathcal{S}(\Phi, X^\alpha)$ and $h(\Phi, K_0) = \Sigma^{b_l}$. Furthermore by Theorem 3.4 we infer that there is an isolating invariant set $K \in \mathcal{S}(\Phi, X^\alpha)$ such that $K_0 \subset K$ and

$$h(\Phi, K) = \begin{cases} \Sigma^{d_k} & \text{if condition (G1) is satisfied;} \\ \Sigma^{d_{k-1}} & \text{if condition (G2) is satisfied.} \end{cases} \quad (3.32)$$

Assumptions (i)–(v) imply that $h(\Phi, K) \neq h(\Phi, K_0)$. Since $h(\Phi, K_0) \neq \bar{0}$, application of Theorems 11.6 and 11.5 from [15] completes the proof. \square

4. APPLICATIONS

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set such that the boundary $\partial\Omega$ is of class C^∞ . Let \mathcal{A} be a differential operator with the Dirichlet boundary conditions given by:

$$\mathcal{A}\bar{u}(x) = - \sum_{i,j=1}^n D_j(a_{ij}(x)D_i\bar{u}(x)) \quad \text{for} \quad \bar{u} \in C^2(\bar{\Omega}),$$

where $a_{ij} = a_{ji} \in C^2(\bar{\Omega})$ for $1 \leq i, j \leq n$ and

$$\sum_{1 \leq i,j \leq n} a_{ij}(x)\xi^i\xi^j \geq c_0|\xi|^2 \quad \text{for} \quad x \in \Omega, \xi \in \mathbb{R}^n,$$

where $c_0 > 0$. Assume that we have a continuous map $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

(E1) there exists $L > 0$ such that for $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$

$$|f(x, s_1, y_1) - f(x, s_2, y_2)| \leq L(|s_1 - s_2| + |y_1 - y_2|),$$

(E2) there exists a constant $m > 0$ such that

$$|f(x, s, y)| \leq m \quad \text{for} \quad x \in \Omega, s \in \mathbb{R}, y \in \mathbb{R}^n.$$

Given $X := L^p(\Omega)$, where $p \geq 1$, define the operator $A_p : X \supset D(A_p) \rightarrow X$ by

$$\begin{aligned} D(A_p) &:= W_0^{2,p}(\Omega) := \text{cl}_{W^{2,p}(\Omega)} \{ \phi \in C^2(\bar{\Omega}) \mid \phi|_{\partial\Omega} = 0 \}, \\ A_p\bar{u} &:= \mathcal{A}\bar{u} \quad \text{for} \quad \bar{u} \in D(A_p). \end{aligned} \quad (4.1)$$

It is well known (see e.g. [5, 17]) that A_p is positively defined sectorial operator on X . Let $X^\alpha := D(A_p^\alpha)$ for $(\alpha \in (0, 1))$ be a fractional space with the norm

$$\|\bar{u}\|_\alpha := \|A_p^\alpha \bar{u}\| \quad \text{for} \quad \bar{u} \in X^\alpha.$$

From now on we assume that

(E3) $p \geq 2n$ and $\alpha \in (3/4, 1)$.

Remark 4.1. (a) Note that A_p satisfies (A1), (A2) and (A3). Since A_p has compact resolvent (see e.g. [5, 17]), the assumption (A1) holds. Assume that $H := L^2(\Omega)$ with the standard inner product and norm

$$\langle \bar{u}, \bar{v} \rangle_{L^2} := \int_{\Omega} \bar{u}(x)\bar{v}(x) dx, \quad \left(\int_{\Omega} |\bar{u}(x)|^2 dx \right)^{1/2} \quad \text{for} \quad \bar{u}, \bar{v} \in H$$

and put $\widehat{A} := A_2$. Then we see that the boundedness of Ω and the fact that $p \geq 2$ imply that there is a continuous embedding $i : L^p(\Omega) \hookrightarrow L^2(\Omega)$ and the assumption (A2) is satisfied. Furthermore we have $D(A_p) \subset D(\widehat{A})$ and $\widehat{A}\bar{u} = A_p\bar{u}$ and $\bar{u} \in D(A_p)$. This shows that $A_p \subset \widehat{A}$ in the sense of the inclusion $i \times i$. Since the operator \widehat{A} is self-adjoint (see e.g. [5]) we see that the assumption (A3) is also satisfied.

(b) Remark 3.1 shows that the spectrum $\sigma(A_p)$ of the operator A_p consists of sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_i < \lambda_{i+1} < \dots \quad \text{for } i \geq 1,$$

and furthermore (λ_i) is finite or $\lambda_i \rightarrow +\infty$ when $i \rightarrow +\infty$.

(c) Note that the following inclusion is continuous

$$X^\alpha \subset C^1(\overline{\Omega}). \quad (4.2)$$

Indeed, according to assumption (E4) we have $\alpha \in (3/4, 1)$ and $p \geq 2n$, and hence $2\alpha - \frac{n}{p} > 1$. Therefore, the assertion is a consequence of [8, Theorem 1.6.1].

(d) If $1 \geq \alpha > \beta \geq 0$ then the inclusion $X^\alpha \subset X^\beta$ is continuous and compact as [8, Theorem 1.4.8] says. \square

According to the point (c) of the above remark we can define a map $F : X^\alpha \rightarrow X$ given, for any $\bar{u} \in X^\alpha$, as

$$F(\bar{u})(x) := f(x, \bar{u}(x), \nabla \bar{u}(x)) \quad \text{for } x \in \Omega. \quad (4.3)$$

We call F the *Niemytzki operator* associated with f and furthermore, it is easy to prove the following lemma

Lemma 4.2. *The map F is well defined, continuous and satisfies assumption (F1). Furthermore there is a constant $K > 0$ such that*

$$\|F(\bar{u})\| \leq K \quad \text{for } \bar{u} \in X^\alpha. \quad (4.4)$$

4.1. Properties of Niemytzki operator. We proceed to study conditions that should satisfy the mapping f in order to the associated Niemytzki operator F meets the introduced earlier geometrical conditions (G1) and (G2). We begin with the following theorem stating that the well-known *Landesman-Lazer* conditions from [10] are actually particular case of the geometrical conditions (G1) and (G2).

Theorem 4.3. *Let $f_+, f_- : \Omega \rightarrow \mathbb{R}$ be continuous functions such that*

$$f_+(x) = \lim_{s \rightarrow +\infty} f(x, s, y) \quad \text{and} \quad f_-(x) = \lim_{s \rightarrow -\infty} f(x, s, y)$$

for $x \in \Omega$, uniformly for $y \in \mathbb{R}^n$ and let $B \subset X_+^\alpha \oplus X_-^\alpha$ be subset, bounded in the norm $\|\cdot\|_\alpha$.

(i) Assume that

$$(LL1) \quad \int_{\{u>0\}} f_+(x)\bar{u}(x) dx + \int_{\{u<0\}} f_-(x)\bar{u}(x) dx > 0$$

for $\bar{u} \in \text{Ker}(\lambda I - A_p) \setminus \{0\}$. Then there exists $R > 0$ such that for any $(\bar{w}, \bar{u}) \in B \times X_0$ with $\|\bar{u}\|_{L^2} \geq R$, one has:

$$\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} > 0.$$

(ii) Assume that

$$(LL2) \quad \int_{\{u>0\}} f_+(x)\bar{u}(x) dx + \int_{\{u<0\}} f_-(x)\bar{u}(x) dx < 0$$

for $\bar{u} \in \text{Ker}(\lambda I - A_p) \setminus \{0\}$. Then there exists $R > 0$ such that for any $(\bar{w}, \bar{u}) \in B \times X_0$ with $\|\bar{u}\|_{L^2} \geq R$, one has:

$$\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} < 0.$$

Proof. The proof of is similar to the proof of Theorem 6.7 from [9]. \square

In the subsequent lemma we show that the geometrical conditions (G1) and (G2) are also consequences of *the strong resonance conditions* from [2], [18].

Theorem 4.4. Assume that there is a continuous function $f_\infty: \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), such that

$$f_\infty(x) = \lim_{|s| \rightarrow +\infty} f(x, s, y) \cdot s$$

for $x \in \Omega$, uniformly for $y \in \mathbb{R}^n$ ad let $B \subset X_+^\alpha \oplus X_-^\alpha$ be a subset, bounded in the norm $\|\cdot\|_\alpha$.

(i) If the following condition holds

$$(SR1) \quad \begin{cases} \text{there is } h \in L^1(\Omega) \text{ such that} \\ f(x, s, y) \cdot s \geq h(x) \text{ for } (x, s, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \text{ and} \\ \int_\Omega f_\infty(x) dx > 0, \end{cases}$$

then there exists $R > 0$ such that for $(\bar{v}, \bar{u}) \in B \times X_0$ with $\|\bar{u}\|_{L^2} \geq R$

$$\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} > 0.$$

(ii) If the following condition holds

$$(SR2) \quad \begin{cases} \text{there is a function } h \in L^1(\Omega) \text{ such that} \\ f(x, s, y) \cdot s \leq h(x) \text{ for } (x, s, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \text{ and} \\ \int_\Omega f_\infty(x) dx < 0, \end{cases}$$

then there exists $R > 0$ such that for $(\bar{w}, \bar{u}) \in B \times X_0$ with $\|\bar{u}\|_{L^2} \geq R$

$$\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} < 0.$$

Proof. The proof of is similar to the proof of Theorem 6.9 from [9]. \square

4.2. Existence of connecting orbits. We shall consider parabolic equations of the form

$$u_t(t, x) = -\mathcal{A}u(t, x) + \lambda u(t, x) + f(x, u(t, x), \nabla u(t, x)), \quad t > 0, \quad x \in \Omega. \quad (4.5)$$

This equation may be written in the abstract form as

$$\dot{u}(t) = -A_p u(t) + \lambda u(t) + F(u(t)), \quad t > 0. \quad (4.6)$$

If $J \subset \mathbb{R}$ is an interval, then we say that $u : J \rightarrow X^\alpha$ is a solution of (4.5), if u is a mild solution of (4.6). From Lemma 4.2 it follows that F satisfies (F1), (F2). Let $\Phi : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ be a semiflow associated with (4.6) defined by

$$\Phi(t, \bar{u}) := u(t; \bar{u}) \quad \text{for } t \in [0, +\infty), \bar{u} \in X^\alpha.$$

From now on we will also assume that

(E4) $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a map of class C^1 such that there is a constant $\nu \in \mathbb{R}$ such that $\nu = D_s f(x, 0, 0)$ for $x \in \Omega$. Furthermore $f(x, 0, 0) = 0$ and $D_y f(x, 0, 0) = 0$ for $x \in \Omega$.

Remark 4.5. By assumption (E4) and (4.2) and one can easily prove that F is differentiable at 0 and its derivative $DF(0) \in L(X^\alpha, X)$ is of the form

$$DF(0)[\bar{u}] = \nu \bar{u} \quad \text{for } \bar{u} \in X^\alpha,$$

and hence assumption (F4) is satisfied.

We start with the following *criterion with Landesman-Lazer conditions*

Theorem 4.6. Suppose that $f_+, f_- : \Omega \rightarrow \mathbb{R}$ are continuous functions such that

$$f_+(x) = \lim_{s \rightarrow +\infty} f(x, s, y) \quad \text{and} \quad f_-(x) = \lim_{s \rightarrow -\infty} f(x, s, y)$$

for $x \in \Omega$, uniformly for $y \in \mathbb{R}^n$. If $\lambda = \lambda_k$ for some $k \geq 1$, then there is a full compact nonzero solution $u : \mathbb{R} \rightarrow X^\alpha$ for (4.5) such that either

$$\lim_{t \rightarrow -\infty} u(t) = 0 \quad \text{or} \quad \lim_{t \rightarrow +\infty} u(t) = 0,$$

provided one of the following conditions is satisfied:

- (i) (LL1) is satisfied holds and $\lambda_l < \lambda + \nu < \lambda_{l+1}$ where $\lambda_l \neq \lambda$;
- (ii) (LL1) is satisfied and $\lambda + \nu < \lambda_1$;
- (iii) (LL2) is satisfied, $\lambda_{l-1} < \lambda + \nu < \lambda_l$ and $\lambda \neq \lambda_l$, where $l \geq 2$;
- (iv) (LL2) is satisfied, $\lambda + \nu < \lambda_1$ and $\lambda \neq \lambda_1$.

Proof. By Remark 4.1 (a) and Remark 4.5, we deduce that assumptions (A1), (A2), (A3) and (F4) are satisfied. Furthermore, from Theorem 4.3 it follows that, condition (LL1) implies condition (G1) and that condition (LL2) implies condition (G2). Therefore Theorem 3.6 completes the proof. \square

Now we proceed to the following *criterion with strong resonance conditions*.

Theorem 4.7. Assume that $\Omega \subset \mathbb{R}^n$ where $n \geq 3$, is an open bounded set and let $f_\infty : \overline{\Omega} \rightarrow \mathbb{R}$ be a continuous function such that

$$f_\infty(x) = \lim_{|s| \rightarrow +\infty} f(x, s, y) \cdot s \quad \text{for } x \in \Omega, \text{ uniformly for } y \in \mathbb{R}^n.$$

Given $\lambda = \lambda_k$ for $k \geq 1$, there is a full compact nonzero solution $u : \mathbb{R} \rightarrow X^\alpha$ of equation (4.5) such that either

$$\lim_{t \rightarrow -\infty} u(t) = 0 \quad \text{or} \quad \lim_{t \rightarrow +\infty} u(t) = 0,$$

provided one of the following conditions is satisfied:

- (i) (SR1) is satisfied and $\lambda_l < \lambda + \nu < \lambda_{l+1}$ where $\lambda_l \neq \lambda$;
- (ii) (SR1) is satisfied and $\lambda + \nu < \lambda_1$;
- (iii) (SR2) is satisfied and $\lambda_{l-1} < \lambda + \nu < \lambda_l$ where $\lambda \neq \lambda_l$, $l \geq 2$;
- (iv) (SR2) is satisfied and $\lambda + \nu < \lambda_1$.

Proof. Similarly as before, Remark 4.1 (a) and Remark 4.5 imply that assumptions (A1), (A2), (A3) and (F4) are satisfied. Furthermore, from Theorem 4.4 it follows that condition (SR1) implies (G1) and that (SR2) implies condition (G2). Therefore the proof is completed after application of Theorem 3.6. \square

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
 NICOLAUS COPERNICUS UNIVERSITY
 CHOPINA 12/18, 87-100 TORUŃ, POLAND
 E-mail address: pkokocki@mat.umk.pl